# DETERMINATION OF BOUNDARY THERMAL REGIME FROM THE SOLUTION OF AN INVERSE HEAT-CONDUCTION PROBLEM

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We consider control schemes for the solution of a one-dimensional inverse heat-conduction problem in a region with moving boundaries. We investigate the computational aspects of the construction of effective algorithms.

Principles of Solution of the Inverse Problem. We consider the general formulation of a one-dimensional inverse problem of type I for a heat-conduction equation with constant coefficients. We seek heat fluxes or temperatures on two moving boundaries of the body according to known temperature dependencies at two interior points. The coordinates of these points can vary in time. Their laws of motion $X_2(\tau)$ and $X_3(\tau)$ and also the laws of motion of the external boundaries $X_1(\tau)$ and $X_4(\tau)$ are known, and are defined by continuous differential functions. Below, for brevity of the discussion, we will consider mainly the problem of determining the heat fluxes. The solution of the similar problem of the temperatures on the surfaces can be obtained (following the considerations given below) in terms of the double-layer thermal potentials. If it is required to establish simultaneously the heat fluxes and the temperatures on the boundaries, then based on the heat fluxes that have been obtained, solving the direct heat-conduction problem, we can find the boundary temperatures. Finally, we assume the initial temperature distribution to be equal to zero, which, however, does not lead to any loss in generality of the formulation of the problem, since, we can always first reduce the initial condition to zero.

Thus, we have the following problem:

$$\frac{\partial\theta}{\partial\tau} = a\frac{\partial^2\theta}{\partial x^2},\quad X_1(\tau)<x<X_4(\tau),\quad \tau>0, \tag{1}$$

$$\left.\begin{aligned}
&\theta(x,\ 0)=0,\\
&-\lambda_0\frac{\partial\theta(X_1(\tau),\ \tau)}{\partial x}=q_1(\tau),\\
&-\lambda_0\frac{\partial\theta(X_4(\tau),\ \tau)}{\partial x}=q_4(\tau),\\
&\theta(X_2(\tau),\ \tau)=f_2(\tau),\\
&\theta(X_3(\tau),\tau)=f_3(\tau),
\end{aligned}\right\} \tag{2}$$

where $f_2(\tau)$, $f(\nu)$ are known functions, and $q_1(\tau)$, $q_4(\tau)$ are unknown functions.

As $\theta$ we can use the model temperature, defined by the Kirchhoff transformation $\theta = 1/\lambda_0\int\limits_0^T\lambda(T)\,dT$.

We introduce into the discussion the single-layer potentials for the heat-conduction equation. Then the function $\theta(x,\tau)$ can be represented in the form

$$\theta(x,\ \tau)=\int\limits_0^\tau \nu_1(\xi)K(x,\ X_1(\xi);\ \tau,\ \xi)\,d\xi+\int\limits_0^\tau \nu_4(\xi)K(x,\ X_4(\xi);\ \tau,\ \xi)\,d\xi,$$

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where $\nu_1$, $\nu_4$ are known functions, and $K(x, X_j(\xi); \tau, \xi) = (1/2\sqrt{a/\pi(\tau-\xi)} \exp[-(x-X_j(\xi))^2/4a(\tau-\xi)]$.

Taking into account the jump conditions for the derivatives of the thermal potentials, we write expressions for determining the two heat fluxes:

$$q_1(\tau) = \frac{\lambda_0}{2}\left\{\frac{\nu_1(\tau)}{2} + \int_0^\tau \nu_1(\xi) V(X_1(\tau), X_1(\xi); \tau, \xi)\, d\xi + \int_0^\tau \nu_4(\xi) V(X_1(\tau), X_4(\xi); \tau, \xi)\, d\xi\right\},$$

$$q_4(\tau) = \frac{\lambda_0}{2}\left\{-\frac{\nu_4(\tau)}{2} + \int_0^\tau \nu_1(\xi) V(X_4(\tau), X_1(\xi); \tau, \xi)\, d\xi + \int_0^\tau \nu_4(\xi) V(X_4(\tau), X_4(\xi); \tau, \xi)\, d\xi\right\},$$

where

$$V(X_i(\tau), X_j(\xi); \tau, \xi) = \frac{X_i(\tau) - X_j(\xi)}{2\sqrt{a\pi(\tau-\xi)^3}} \exp\left[-\frac{(X_i(\tau) - X_j(\xi))^2}{4a(\tau-\xi)}\right].$$

The densities of the thermal potentials should satisfy the following Volterra integral equation of the first kind;

$$A\nu \equiv \int_0^\tau K(\tau, \xi)\, \nu(\xi)\, d\xi = f(\tau), \tag{3}$$

where we have introduced the two-component vector-functions

$$\nu(\xi) = \begin{bmatrix} \nu_1(\xi) \\ \nu_4(\xi) \end{bmatrix}, \quad f(\tau) = \begin{bmatrix} f_2(\tau) \\ f_3(\tau) \end{bmatrix}$$

and the matrix of the kernels

$$K(\tau, \xi) = \begin{bmatrix} K(X_2(\tau), X_1(\xi); \tau, \xi) & K(X_2(\tau), X_4(\xi); \tau, \xi) \\ K(X_3(\tau), X_1(\xi); \tau, \xi) & K(X_3(\tau), X_4(\xi); \tau, \xi) \end{bmatrix}.$$

The present problem is incorrect (unstable against small perturbations in the right side); therefore, we must seek its solution on the basis of Tikhonov's regularization method [1].

We construct a regularization function in the form

$$\Phi[\nu, \alpha] = \|A\nu - f\|^2 + \alpha\|\nu\|^2, \tag{4}$$

where

$$\|A\nu - f\|^2 = \int_0^{\tau_m} \left[f_2(\tau) - \int_0^\tau \nu_1(\xi) K(X_2(\tau), X_1(\xi); \tau, \xi)\, d\xi - \int_0^\tau \nu_4(\xi) K(X_2(\tau), X_4(\xi); \tau, \xi)\, d\xi\right]^2 d\tau + \int_0^{\tau_m} \left[f_3(\tau) - \int_0^\tau \nu_1(\xi) K(X_3(\tau), X_1(\xi); \tau, \xi)\, d\xi - \int_0^\tau \nu_4(\xi) K(X_3(\tau), X_4(\xi); \tau, \xi)\, d\xi\right]^2 d\tau,$$

$$\|\nu\|^2 = \int_0^{\tau_m} \left[\nu_1'(\tau) + \nu_4'(\tau)\right]^2 d\tau.$$

The solution of the problem of the minimization of the functional (4) with respect to $\nu_1(\tau)$ and $\nu_4(\tau)$ can be reduced to the solution of the two corresponding Euler equations, depending on the regularization parameter $\alpha$. By appropriate selection of $\alpha$, we can achieve stability and the necessary accuracy of the solution. Thus, the general methodology of regularization of the inverse heat-conduction problem with two unknown boundary conditions does not differ principally from the case of a single unknown condition [2,3]. However, in the computational plan, the problem becomes more complex and laborious and we therefore also consider other possible approaches to its solution.

If, above, for obtaining Eq. (3) we considered the region $D\{X_1(\tau) \le x \le X_4(\tau),\ \tau > 0\}$, then we now consider the two regions $D_1\{X_1(\tau) \le x \le X_2(\tau),\ \tau > 0\}$ and $D_2\{X_2(\tau) \le x \le X_3(\tau),\ \tau > 0\}$ with the common boundary $X_2(\tau)$, on which the following coupling conditions are satisfied:

$$\theta_{D_1}(X_2(\tau),\ \tau) = \theta_{D_2}(X_2(\tau),\ \tau),$$

$$\frac{\partial\theta_{D_1}(X_2(\tau),\ \tau)}{\partial x} = \frac{\partial\theta_{D_2}(X_2(\tau),\ \tau)}{\partial x}.$$

Similarly we can consider the regions $D_2$ and $D_3\{X_3(\tau) \le x \le X_4(\tau),\ \tau > 0\}$, bordering on the boundary $X_3(\tau)$.

The heat flux $q_2(\tau)$ is found from a solution of the direct heat-conduction problem in the region $D_2$. In this case we can use the double-layer thermal potentials [4]. In the present case, applying the approximation described in [5] for the integral terms (in the system of Volterra equations of the second kind, written for the determination of the densities of the potentials), we succeed in reducing this problem to the successive solution, for each step in time, of a system of two linear algebraic equations. As was shown by numerical experiments, such a method proves to be very effective with respect to the accuracy of the results obtained and the expenditure of calculation time.

It is necessary to obtain the unknown densities $\nu_1$ and $\nu_2$, corresponding to the boundaries $X_1(\tau)$ and $X_2(\tau)$ of region $D_1$ from the solution of the following system of integral equations:

$$\int_0^\tau \nu_1(\xi) K(X_2(\tau),\ X_1(\xi);\ \tau,\ \xi)\, d\xi + \int_0^\tau \nu_2(\xi) K(X_2(\tau),\ X_2(\xi);\ \tau,\ \xi)\, d\xi = f_2(\tau), \tag{5}$$

$$-\frac{\nu_2(\tau)}{2} + \int_0^\tau \nu_2(\xi) V(X_2(\tau),\ X_2(\xi);\ \tau,\ \xi)\, d\xi + \int_0^\tau \nu_1(\xi) V(X_2(\tau),\ X_1(\xi);\ \tau,\ \xi)\, d\xi = \frac{2q_2(\tau)}{\lambda_0}. \tag{6}$$

Equation (5) is an equation of the first kind in $\nu_1$ and $\nu_2$; however, it can be reduced to an equation of the second kind in $\nu_2$.

Following the method of Gol'mgren, used in [6], we multiply both sides of Eq. (5) by $(z-\tau)^{1/2}$, integrate over $\tau$ from 0 to z, reverse the order of integration in the integrals on the left side, and, finally, differentiate the equation obtained with respect to z with subsequent replacement of z by $\tau$. Omitting all the calculations related to these transformations, we write the final result

$$\frac{\sqrt{\pi a}}{2}\nu_2(\tau) + \int_0^\tau \nu_1(\xi) W_1(\xi,\ \tau)\, d\xi + \int_0^\tau \nu_2(\xi) W_2(\xi,\ \tau)\, d\xi = \frac{1}{\sqrt{\tau}} f_2(\tau) - \frac{1}{2}\int_0^\tau \frac{f_2(\xi) - f_2(\tau)}{\sqrt{(\tau-\xi)^3}}\, d\xi, \tag{7}$$

where

$$W_l(\xi,\ \tau) = \frac{1}{4}\sqrt{\frac{a}{\pi}} \times \int_\xi^\tau \frac{\exp\left[-\frac{(X_2(\tau) - X_l(\xi))^2}{4a(\tau-\xi)}\right] - \exp\left[-\frac{(X_2(z) - X_l(\xi))^2}{4a(z-\xi)}\right]}{(\tau-z)^{3/2}(z-\xi)^{1/2}}\, dz.$$

We note that in the derivation of (7) we have used the assumption of Holder continuity of the function $f_2(\tau)$ with exponent $(1+\beta)/2$ $(0 < \beta \le 1)$.

The kernels $W_l(\xi, \tau)$ have a weak singularity of the form [6]

$$|W(\xi,\ \tau)| \leqslant M(\tau-\xi)^{-1+\frac{\varkappa}{2}},\ M>0,\ 0<\varkappa<1,$$

which indicates the existence of the integrals on the left side of (7) and the possibility of representing them in finite-difference form.

Thus, in the given formulation of the inverse heat-conduction problem it is necessary to regularize the problem of the determination of the single unknown density $\nu_1(\tau)$.

We exclude the most typical (in practice, the solutions of inverse heat-conduction problems with constant coefficients) case in which the points with known temperatures are not displaced during heating, i.e., $X_2(\tau)$ = const, $X_3(\tau)$ = const. Then the problem (5), (6) reduces to a solution of the integral equation

$$\frac{1}{\sqrt{\pi a}}\int_0^\tau \nu_1(\xi)W_1(\xi,\ \tau)\,d\xi+\int_0^\tau \nu_1(\xi)V\left(X_2(\tau),\ X_1(\xi);\ \tau,\ \xi\right)d\xi$$

$$=\frac{2q_2(\tau)}{\lambda_0}+\frac{1}{\sqrt{\pi a\tau}}f_2(\tau)-\frac{1}{2\sqrt{\pi a}}\int_0^\tau \frac{f_2(\xi)-f_2(\tau)}{\sqrt{(\tau-\xi)^3}}\,d\xi. \tag{8}$$

Equation (8) can be approximated in the form of the sum (taking account of the singularity of $W_1(\xi, \tau)$)

$$A\nu_1\equiv\sum_{i=1}^{n} p_i k_{in}\nu_{1i}=F_n;\ n=1,\ 2,\ \ldots,\ m, \tag{9}$$

where the $p_i$ are the weighting factors, which depend on the quadratic formula that is used.

Regularization of (9) according to the Tikhonov method leads to the equation

$$(A^{\mathrm{T}}A+\alpha C)\,\nu_1=A^{\mathrm{T}}F,$$

where C is a positive-definite symmetric matrix, the form of which is determined by the order of regularization (see, e.g., (14) and (15) below).

The approach considered above is equivalent to the reduction of the inverse problem to a Cauchy problem with conditions $f_2(\tau)$ and $q_2(\tau)$, assigned for $x = X_2(\tau)$. A specific shortcoming of such a representation consists of the need to first calculate the heat flux $q_2(\tau)$. If the errors in the initial data $f_2(\tau)$ and $f_3(\tau)$ can usually be assumed to be independent of each other, then, in the transition to the Cauchy problem, the errors in the function $q_2(\tau)$, obtained from a solution of the direct heat-conduction problem, to a considerable extent, are determined by the errors in the assignment of $f_2(\tau)$. "Synchronization" of the errors in $f_2$ and $q_2$ can worsen the accuracy of the solution of the inverse heat-conduction problem.

We now discuss a method of reducing the boundary thermal regimes $q_1(\tau)$ and $q_4(\tau)$ to the general formulation (1), (2). It is based on two transformations, which enable us to formulate, instead of (1), (2), a problem with fixed boundaries of the input data and fixed boundaries of the unknown functions. The first transformation is a conversion to the new input data $\theta_2(\tau)$ and $\theta_3(\tau)$ for some straight lines $x_2$ = const and $x_3$ = const, which are obtained from a solution of the direct heat-conduction problem in the region $D_2$. For definiteness, we can assume, for example, $x_2 = X_{2\max}$ and $x_3 = X_{3\min}$. If the input functions are given at fixed points, then naturally such a transformation is not required. The second transformation is the final conversion to the rectangular regions $D_1'\{0 \le x \le x_2,\ \tau > 0\}$ and $D_3'\{x_3 \le x \le X_{4\max},\ \tau > 0\}$, enclosing the regions $D_1$ and $D_3$ $(D_1 \subseteq D_1',\ D_3 \subseteq D_3')$.

We now solve the inverse problems of the determination of the fictitious temperatures $\theta_{f1}$ and $\theta_{f2}$, corresponding to the boundaries $x = 0$ and $x = X_{4\max}$. This can be done sufficiently easily in terms of the Duhamel integral with the use of the principle of superposition of solutions. For example, for $\theta_{f1}$ we have the integral equation

$$\int_0^\tau \theta_{f1}(\xi)\,\frac{\partial\vartheta(x_3-x_2,\ \tau-\xi)}{\partial\tau}\,d\xi=g(\tau), \tag{10}$$

where

$$g(\tau)=\theta_2(\tau)-\int_0^\tau \theta_3(\xi)\frac{\partial\vartheta(x_2,\tau-\xi)}{\partial\tau}d\xi;$$

$\vartheta(x, \tau-\xi)$ is a solution of the first boundary-value problem in the region $D_1' \cup D_2'$ for the condition of zero temperature on one boundary and unity temperature on the other.

Thus, the considered transformation of the initial formulation of the problem allowed us to obtain a fundamental integral equation (10) in a form such that its solution is completely confined within the limits of the regularization schemes constructed in [2, 3].

The last step in determining the unknown conditions on the boundaries $X_1(\tau)$ and $X_4(\tau)$ consists of solving the direct heat-conduction problems, respectively, in $D_1'$ and $D_3'$. These problems do not present any fundamental difficulties. Their method of solution can be based on the Duhamel principle or on the theory of thermal potentials.

The approaches analyzed above for the solution of the inverse heat-conduction problem in a homogeneous body can also be extended to multilayered bodies with thermal-conductivity coefficients that are piecewise-constant over the coordinate.

## Some Computational Aspects of the Regularization of the Solutions of the Inverse Problems

In many cases, with the use of regularization schemes for solution of the inverse heat-conduction problems, it becomes necessary to completely automate the process of selecting reasonable, i.e., optimal in a certain sense, approximations to the unknown boundary functions (selection of the regularization parameter). The determination of the approximations from the condition of internal convergence of the regularization solutions (the quasioptimal-parameter method [7]) usually requires a qualitative analysis of the obtained results. In the present case we use certain a priori information on the expected solution and the assumed character of the exact input function. Furthermore, as was noted in [3, 8], it is advisable to use the quasioptimal-parameter method in combination with other methods for determining the optimal approximations. All of this significantly hinders the automation of the process being considered. At the same time, the criteria for selecting the regularization parameter, which are based on V. A. Morozov's discrepancy principle [9, 10], allow us to construct an automatic search of the best approximations.

An effective algorithm for solving the given problem was proposed in [11]. We consider its application to the solution of inverse heat-conduction problems in the formulation of [2]. It is required to find an m-component vector u (heat flux, temperature, or density of the thermal potential on the boundary of the body) from a solution of the following system of algebraic equations with lower triangular matrix:

$$Au\equiv\sum_{i=1}^{m}\varphi_i^n u_i=f_n,\ n=1,\ 2,\ \ldots,\ m. \tag{11}$$

The coefficients $\varphi_i^n$ are determined as a function of the inverse problem being solved (for a semiinfinite body and for a plate they are given in [3]).

In agreement with the regularization discrepancy principle for the condition of obtaining the unknown solution with minimum Euclidean norm of the first differences, the problem (11) reduces to the solution of the parametric system of linear algebraic equations

$$(B+\alpha C)u=g \tag{12}$$

jointly with the condition for the selection of $\alpha$

$$\rho(\alpha)\equiv\Big[\sum_{n=1}^{m}\Big(\sum_{i=1}^{n}\varphi_i^n u_{i\alpha}-f_n\Big)^2\Big]^{\frac{1}{2}}=\delta, \tag{13}$$

where

$B=\left\{b_l^k=\sum_{n=l}^{m}\varphi_k^n\varphi_l^n\right\},\ l=1,\ 2,\ \ldots,\ m,\ k=1,\ 2,\ \ldots,\ m;$ $\alpha$ is the regularization parameter;

$$C=\begin{bmatrix} 1 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix};$$

$$g=\Big\{g_k=\sum_{n=k}^{m}\varphi_k^n f_n\Big\},\ k=2,\ 3,\ \ldots,\ m-1;$$

$$g_1=\sum_{n=1}^{m}\varphi_1^n f_n-\alpha c_1 \quad (c_1=u'(0));$$

$$g_m=\varphi_m^m f_m-\alpha c_2 \quad (c_2=u'(\tau_m)). \tag{14}$$

If we formulate a problem about the minimization of the norm of the second differences, then for the natural boundary conditions $u''(0)=u''(\tau_m)=u'''(0)=u'''(\tau_m)$ we have

$$C=\begin{bmatrix} 1 & -2 & 1 & & & & \\ -2 & 5 & -4 & 1 & & & \\ 1 & -4 & 6 & -4 & 1 & & \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & 1 & -4 & 6 & -4 & 1 \\ & & & 1 & -4 & 5 & -2 \\ & & & & 1 & -2 & 1 \end{bmatrix}. \tag{15}$$

In [11] it was shown that if we replace the discrepancy equation $\|Au_\alpha-f\|_{E_m}=\delta$ by the equation $\|Au_\alpha-f\|^s_{E_m}=\delta^s$, $s\geq -1$, and we consider the latter with respect to $p=1/\alpha$, then for its solution we can apply Newton's method of tangents (owing to the convexity from below of the functions $\rho^s(1/p)=\|Au_{1/p}-f\|^s_{E_m}$). Furthermore, we determine that a high rate of convergence for the process of searching for the roots exists for $s=-1$.

Thus, instead of (13), below, we will solve the equation

$$F(p)\equiv\Big[\sum_{n=1}^{m}\Big(\sum_{i=1}^{n}\varphi_i^n u_{i1/p}-f_n\Big)^2\Big]^{-1/2}-\delta^{-1}=0. \tag{16}$$

We construct an iterational sequence based on the equation

$$p_{n+1}=p_n-\frac{F(p_n)}{F'(p_n)},$$

where

$$F'(p)=-\frac{1}{\rho^3_{1/p}p^2}\sum_{n=1}^{m}\Big[\Big(\sum_{i=1}^{n}\varphi_i^n u_{i1/p}-f_n\Big)\sum_{i=1}^{n}\varphi_i^n\frac{du_{i1/p}}{d\alpha}\Big]. \tag{17}$$

Having differentiated (12), we obtain a system for the determination of $du_{i1/p}/d\alpha$

$$\Big(B+\frac{1}{p}C\Big)\frac{du_{1/p}}{d\alpha}=b. \tag{18}$$

The column vector b for the matrix C, determined by (14), has the form

$$b_1=-c_1-u_1+u_2,$$

$$b_k=u_{k-1}-2u_k+u_{k+1},\ k=2,\ 3,\ \ldots,\ m-1,$$

$$b_m=-c_2+u_{m-1}-u_m.$$

If the stabilizing term in Eq. (12) corresponds to the matrix (15), then

$$b_1 = u_1 + 2u_2 - u_3,$$
$$b_2 = 2u_1 - 5u_2 + 4u_3 - u_4,$$
$$b_k = -u_{k-2} + 4u_{k-1} - 6u_k + 4u_{k+1} - u_{k+2},\ k = 3,\ 4,\ \ldots,\ m-2,$$
$$b_{m-1} = -u_{m-3} + 4u_{m-2} - 5u_{m-1} + 2u_m,$$
$$b_m = -u_{m-2} + 2u_{m-1} - u_m.$$

The process of approaching $p = p_d$, the root of Eq. (16) (according to the discrepancy), can be started from a certain guaranteed value $p_{min} < p_d$, which either is given a priori or is calculated by using, for example, the estimate of [12]:

$$\frac{1}{p_{\min}} \approx \frac{\delta \sum_{i,n} (\varphi_i^n)^2}{\left(\sum_{n=1}^{m} f_n^2\right)^{1/2} - \delta}.$$

If we assume $p = 0$ for the initial value, then, as can be shown,

$$F(0) = \|f\|^{-1} - \delta^{-1},$$
$$F'(0) = \frac{1}{\|f\|^3} (A^{\mathrm{T}}f,\ C^{-1}A^{\mathrm{T}}f). \tag{19}$$

We can avoid transformations of the matrix C, and construct a more economical and exact algorithm for solving the above problem if we apply to (12) and (13) the transformations proposed in [13]. In this case, instead of (12), (13), (17), (18) and (19) we have, respectively,

$$\left.\begin{aligned} &\left(D^{\mathrm{T}}D + \frac{1}{p}E\right) u_{1/p} = D^{\mathrm{T}}g, \\ &\rho_{1/p} = \|Du_{1/p} - g\|_{E_m}, \\ &F'(p) = \frac{-\left(u_{1/p},\ \frac{du_{1/p}}{dp}\right)}{\rho_{1/p}^3 p^3}, \end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned} &\left(D^{\mathrm{T}}D + \frac{1}{p}E\right) \frac{du_{1/p}}{dp} = -u_{1/p}, \\ &F'(0) = \frac{1}{\rho_{1/p}^3} \|D^{\mathrm{T}}g\|_{E_m}^2. \end{aligned}\right\} \tag{21}$$

Thus, in the calculation of the usual value of p, we solve the systems (12) and (18) or (20) and (21), which differ only in their right-hand sides. For the purpose of economy of computational time we solve the systems (20) and (21) by the pivotal method.

Note that for the coefficients $\varphi_i^n$ and $b_l^k$, the following equations are satisfied:

$$\varphi_i^n = \varphi_{m-n+i}^m,\ \varphi_i^n = \varphi_n^i,\ b_l^k = b_{m-k+l}^m,\ b_l^k = b_k^l(\dot{X}=0).$$

This allows us at once to calculate the m values of the coefficients $\varphi_i^m$ and the m values of the coefficients $b_l^m$, which saves a good deal of machine time.

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## NOTATION

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|---|---|
| A | is an operator or matrix; |
| B, C, D | are matrices; |
| $A^T$, $D^T$ | are transposed matrices; |
| $C^{-1}$ | is the matrix inverse to C; |
| E | is the unit matrix; |
| $E_m$ | is an m-dimensional Euclidean space; |
| $a$ | is the thermal-diffusivity coefficient; |

$f$ is the input data;
$q$ is the heat flux;
$T$, $\theta$ are the temperature and the model temperature;
$u$ is a solution of the integral equation;
$X$ is the coordinate of the moving boundary;
$x$ is a coordinate;
$\alpha$ is a parameter of the regularization;
$\delta$ is the error in the input data;
$\lambda$ is the thermal-conductivity coefficient;
$\nu$ is the single-layer thermal-potential density;
$\tau$ is the time;
$\tau_m$ is the right-hand limiting value of the time interval;
$\|\cdot\|$ is the norm.

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